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Some Aspects of Best n-Convex Approximation

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An n-convex function is one whose nth order divided differences are nonnegative. Thus a 1-convex function is nondecreasing and a 2-convex function is convex in the classical sense. A function f is n-concave if $-f$ is n-convex. We consider best uniform approximation by n-convex and n-concave functions and, by considering alternation properties of the error function, we prove a variety of results, including characterizations of functions with best n-convex or n-concave approximations in Π_{n-1}, a sufficient condition for best n-convex approximation, and a uniqueness result. Several examples are given.

INTRODUCTION

A real-valued function f is called n-convex if its nth order divided differences $[x_0, ..., x_n]f$ are nonnegative for all $x_0 < \cdots < x_n$. Thus a 1-convex function is nondecreasing and a 2-convex function is convex in the usual sense. An n-convex function f need not be n-times differentiable, however if $f^{(n)}$ exists then f is n-convex iff $f^{(n)} \geqslant 0$. A function f is n-concave if $-f$ is n-convex. The study of n-convex functions was initiated by Hopf in his dissertation [3] in 1926, and they were the subject of a monograph by Popoviciu [10] in 1944. The book [11] by Roberts and Varberg also contains an introduction to n-convex functions, as well as to other forms of generalized convexity.

The subject of n-convexity may be viewed from the broader perspective of WT-spaces and generalized convexity, a subject that has been intensively studied by Karlin and Studdden [4] and others, including the author [13–16]. Many of the properties of n-convex functions are shared by a larger class of generalized convex functions and, indeed, can be demonstrated by appealing to this general theory.

In this paper we will assume no such knowledge, however we note that, in the terminology of those papers, a function f is n-convex if it is generalized convex with respect to the T-system $\{1, x, ..., x^{n-1}\}$, so that

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either f is a polynomial of degree at most $n-1$ or else $\{1, x, ..., x^{n-1}, f\}$ is a WT-system.

This paper deals with best uniform approximation of continuous functions from the convex cone of n-convex functions. A convex function g will be called a *best n-convex approximation* to f if

$$\sup |f(x)-g(x)| = \inf\{\sup |f(x)-\tilde{g}(x)| : \tilde{g} \text{ is } n\text{-convex}\}.$$

There has been since the mid-sixties continued interest in so-called shape preserving approximations—approximations that preserve properties of monotonicity or convexity, or that satisfy certain restrictive derivative conditions (see, e.g., [6, 7, 12]). We believe that this subject can be treated fruitfully in the more general framework of n-convex approximation, and this paper is a start in that direction.

For the most part, the results of this paper deal with qualitative aspects of best n-convex approximation, and existence is not considered. The interested reader is referred to [17], where it is proved that every continuous function has a continuous best n-convex approximation.

1

We will use the following conventions and notation throughout this paper:

(1) All functions are assumed to be in $C[a, b]$, i.e., continuous on the compact interval $[a, b]$, unless otherwise noted.

(2) $\|f\|$ denotes the uniform norm of f, i.e., $\|f\| = \sup\{|f(x)| : x \in [a, b]\}$.

(3) The term "best approximation" will always connote "best uniform approximation," unless otherwise stated. Thus g is a *best n-convex approximation* (bna) to f if

$$\|f-g\| = \inf\{\|f-\tilde{g}\| : \tilde{g} \text{ is } n\text{-convex on } [a, b]\}.$$

(4) Π_k denotes the linear space of polynomials of degree at most k.

(5) For a given (continuous) function, p_{n-1} denotes its unique best approximation from Π_{n-1} and p_n is its best approximation from Π_n. The leading coefficient of p_n is denoted by a_n.

(1) Definition. An *alternant of length k* for a function f is a set of points $a \leqslant x_1 < \cdots < x_k \leqslant b$ such that

$$f(x_i) f(x_{i+1}) < 0 \qquad (i = 1, ..., k-1)$$

and

$$|f(x_i)| = \|f\| \qquad (i=1, ..., k).$$

If $f(x_k) > 0$ we call it a *positive alternant*, otherwise it is a *negative alternant.*

We recall [1] that $p_{n-1} \in \Pi_{n-1}$ is a best uniform approximation of $f \in C[a, b]$ iff $f - p_{n-1}$ has an alternant of length at least $n+1$. Best polynomial approximations are unique.

In the following lemma, and henceforth, a zero of multiplicity two for a function f on $[a, b]$ is a zero in (a, b) at which f does not change sign. Lemma (2) is a compilation (see [10, 11, 13]) of various results about n-convex functions.

(2) LEMMA. *If g is n-convex on $[a, b]$ then*

(a) *g has at most n sign changes in (a, b),*

(b) *g has at most n isolated zeros (counting multiplicity up to two),*

(c) *if g has more than n zeros (counting multiplicity up to two) then g vanishes on a subinterval of $[a, b]$ and is nonzero elsewhere in $[a, b]$,*

(d) *$[x_0, ..., x_n]\, g = 0$ for some $x_0 < \cdots < x_n$ iff $g \in \Pi_{n-1}$ on $[x_0, x_n]$, and*

(e) *$g^{(n-2)}$ is continuous and convex in (a, b).*

(3) DEFINITION. An *oscillation of length k* for a function f is a set of points $x_1 < \cdots < x_k$ such that $\varepsilon(-1)^i(f(x_{i+1}) - f(x_i)) > 0$ $(i=1, ..., k-1)$ for $\varepsilon = \pm 1$. If $f(x_k) - f(x_{k-1}) > 0$ we call it a *positive oscillation*, otherwise it is a *negative oscillation.*

The following lemma is fundamental to the further results of this paper.

(4) LEMMA. *No n-convex function has a negative oscillation of length $n+1$.*

Proof. For $n=1$ and $n=2$ this is obvious, so assume $n \geqslant 3$. In this case, if g is n-convex then g is differentiable in (a, b) and g' is $(n-1)$-convex in (a, b). Suppose that g has a negative oscillation $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$, i.e., $(-1)^{n-i}(g(x_{i+1}) - g(x_i)) < 0$ $(i=1, ..., n)$. By the mean value theorem there are points $x_i < y_i < x_{i+1}$ such that $(-1)^{n-i} g'(y_i) < 0$ $(i=1, ..., n)$. But then

$$[y_1, ...\, y_n]\, g' = \sum_{i=1}^{n} \frac{g'(y_i)}{\prod_{j \neq i} (y_i - y_j)} < 0$$

since $\operatorname{sgn} \prod_{j=1, j \neq i}^{n} (y_i - y_j) = (-1)^{n-i}$. This contradicts the $(n-1)$-convexity of g' in (a, b). ▮

The following theorem follows directly from (4) and the statement about sign changes in (2).

(5) THEOREM. *f is n-convex iff for every $p \in \Pi_{n-1}$, $f - p$ has no negative oscillation of length $n+1$.*

(6) DEFINITION. For $f \in [a, b)$ we define $\operatorname{crit}(f) = \{x \in [a, b]: |f(x)| = \|f\|\}$.

(7) LEMMA. *If g is bna to f then $f - g$ has an alternant of length at least $n+1$.*

Proof. If $f - g$ has at most n alternation points then we could define a polynomial $p \in \Pi_{n-1}$ such that $\operatorname{sgn} p(x) = \operatorname{sgn}(f(x) - g(x))$ for $x \in \operatorname{crit}(f - g)$. Then, for small enough $\gamma > 0$, $f - g - \gamma p$ would have smaller norm than $f - g$ and thus $g + \gamma p$ would be a better n-convex approximation to f. ▮

(8) LEMMA. *There is a negative alternant of length $n+1$ common to all $f - g$ where g is a bna to f.*

Proof. We first prove that the assertion in (8) is valid for a countable collection of bna's. Let $\{g_i\}_{i=1}^{\infty}$ be a bna's to f, set $E(f) := \|f - g_i\|$, and for $\beta_i > 0$ $(i = 1, 2, ...)$ such that $\sum_{i=1}^{\infty} \beta_i = 1$ define $g := \sum_{i=1}^{\infty} \beta_i g_i$. Since $\|g_i\| - \|f\| \leqslant \|f - g_i\| = E(f)$, the sequence $\{\|g_i\|\}$ is uniformly bounded; hence g is the uniform limit of continuous, n-convex functions and thus is continuous and n-convex. Moreover, g is a bna to f since

$$E(f) \leqslant \|f - g\| = \left\| \sum_i \beta_i (f - g_i) \right\| \leqslant \sum_i \beta_i \|f - g_i\| = E(f).$$

We now show that

$$\operatorname{crit}(f - g) \subset \bigcap_{i=1}^{\infty} \operatorname{crit}(f - g_i), \tag{9}$$

from which the assertion for countable collections will follow by applying (7) to $f - g$. Suppose that $x \in \operatorname{crit}(f - g)$ with $f(x) - g(x) = +E(f)$. Then

$$E(f) = f(x) - g(x) = \sum_i \beta_i (f(x) - g_i(x)) \leqslant \sum_i \beta_i E(f) = E(f);$$

hence equality prevails, which is possible only if $f(x)-g_i(x)=E(f)$ $(i=1, 2, \ldots)$, i.e.,

$$x \in \bigcap_i \operatorname{crit}(f-g_i). \tag{10}$$

Similarly, if $f(x)-g(x)=-E(f)$, then $f(x)-g_i(x)=-E(f)$ $(i=1, 2, \ldots)$, so that (10) holds in this case a well. This proves the validity of (9).

Now let $\{g_\alpha\}$ be the collection of all bna's to f and define $C_\alpha=\operatorname{crit}(f-g_\alpha)$. The sets C_α are compact and nonempty. Let $C=\bigcap_\alpha C_\alpha$; then C is closed, and hence

$$C^c=\left(\bigcap_\alpha C_\alpha\right)^c=\bigcup_\alpha C_\alpha^c$$

is an open covering of C^c. By Lindelöf's theorem this can be reduced to a countable subcovering: $C^c=\bigcup_{i=1}^\infty C_{\alpha_i}^c$, and thus $C=\bigcap_{i=1}^\infty C_{\alpha_i}$. As we have shown above, this intersection contains a common negative alternant of length $n+1$, and therefore the lemma is proved. ∎

The following is a sort of "de la Vallée Poussin" theorem for n-convex functions (cf. [1]).

(11) THEOREM. *Suppose that, for some n-concave function h, $f-h$ has a negative oscillation of length $n+1$: $x_1<\cdots<x_{n+1}$, $f(x_j)-h(x_j)=(-1)^{n-j}e_j$, $e_j>0$ $(j=1, \ldots, n+1)$. Then, for every n-convex function g, $\|f-g\| \geqslant \min_j e_j$.*

Proof. Otherwise, for some n-convex function g, $\|f-g\|<\min e_j$; hence $\|f-g\|<\|f-h\|$ and $g-h=(f-h)-(f-g)$ has a negative oscillation of length $n+1$, a contradiction since $g-h$ is n-convex. ∎

(12) COROLLARY. *If for some n-concave function h, $f-h$ has a negative alternant of length $n+1$, then, for every n-convex function g, $\|f-g\| \geqslant \|f-h\|$.*

2

We now prove our first main theorem.

(13) THEOREM. *For $f \in C[a, b]$, the following are equivalent:*

(a) *p_{n-1} is a bna,*

(b) *p_n is n-concave $(a_n \leqslant 0)$, and*

(c) *$f-p_{n-1}$ has a negative alternant of length $n+1$.*

Proof. (a) $\Rightarrow$ (b). We observe that if $a_n \neq 0$ then by the uniqueness of p_n

$$\|f - p_n\| < \|f - p_{n-1}\|. \tag{14}$$

If p_n is not n-concave, i.e., $a_n > 0$, then (14) holds and p_n is a better n-convex approximation than p_{n-1}.

(b) $\Rightarrow$ (c). If $a_n = 0$ then $p_{n-1} = p_n$; thus $f - p_{n-1}$ has an alternant of length $n+2$ and hence it also has a negative alternant of length $n+1$. If $a_n < 0$ then (14) holds; hence $p_n - p_{n-1} = (f - p_{n-1}) - (f - p_n)$ has an oscillation of length $n+1$ at the alternation points of $f - p_{n-1}$, with the same orientation. Since $p_n - p_{n-1}$ is n-concave, by (4) the oscillation (and hence also the alternant) must be negatively oriented.

(c) $\Rightarrow$ (a). This follows from (12) by setting $h := p_{n-1}$. ∎

(15) COROLLARY. *Every continuous function has either a best n-convex approximation in Π_{n-1} or a best n-concave approximation in Π_{n-1}.*

(16) COROLLARY. *Let g be any bna to $f \in C[a, b]$. If $a_n < 0$ then*

(a) $\|f - g\| = \|f - p_{n-1}\|$,

(b) *$f - g$ has a negative alternant of length $n+1$,*

(c) *$f - g$ has no positive alternant of length $n+1$,*

(d) *$f - g$ has no alternant of length $n+2$.*

Proof. As shown in (15), $a_n < 0$ implies that $\|f - g\| = \|f - p_{n-1}\| > \|f - p_n\|$. Now the n-concave function $p_n - g = (f - g) - (f - p_n)$ oscillates where $f - g$ alternates; hence $f - g$ can have no positive alternant of length $n+1$ (and thus no alternant of length $n+2$). However, by (7) $f - g$ has at least one alternant of length $n+1$, which must therefore be negatively oriented. ∎

(17) THEOREM. *For $f \in C[a, b]$, the following are equivalent:*

(a) *p_{n-1} is both a best n-convex approximation and a best n-concave approximation to f,*

(b) $p_{n-1} \equiv p_n$ $(a_n = 0)$,

(c) *$f - p_{n-1}$ has an alternant of length $n+2$,*

(d) *for some n-concave function h, $f - h$ has a negative alternant of length $n+1$, and for some n-convex function g, $f - g$ has a positive alternant of length $n+1$, and*

(e) *$\|f - g\| = \|f - h\|$ for some best n-convex approximation g and some best n-concave approximation h to f.*

Proof. The equivalence of (a) and (b) follows directly from (13) and the analogous statement for n-concave approximation, as does that of (a) and (c). Now (c) $\Rightarrow$ (d) by setting $g := h = p_{n-1}$ and (d) $\Rightarrow$ (e) by applying (12) and its n-concave analog. Finally, (e) $\Rightarrow$ (a) since by (15) $\|f-g\| = \|f-p_{n-1}\|$ or $\|f-h\| = \|f-p_{n-1}\|$, so that all are equal. ∎

(18) THEOREM. *Let $f \in C[-a, a]$. If f is an even function and n is odd or if f is an odd function and n is even then f has both a best n-convex and a best n-concave approximation in Π_{n-1}.*

Proof. Suppose that f is even on $[-a, a]$. Then p_n is even as well since $\frac{1}{2}(p_n(x)+p_n(-x))$ is also a best approximation to $f(x) = \frac{1}{2}(f(x)+f(-x))$, and hence by uniqueness it equals $p_n(x)$. If n is odd it follows that $p_n \in \Pi_{n-1}$, i.e., $p_n = p_{n-1}$, and we may use (17). We proceed in a similar fashion if f is odd and n is even. ∎

(19) EXAMPLE. Let $f(x) = |x|$ on $[-1, 1]$. Since f is an even function its best approximation from Π_3 is also even and hence is in Π_2. Now the best quadratic approximation to x on $[0, 1]$ is gotten by considering the Chebyshev polynomial $T_2(x) = 2x^2 - 1$ on $[-1, 1]$ transformed for $[0, 1]$:

$$x = 2y - 1 \Rightarrow T_2(y) = y^2 - y - \tfrac{1}{8}, \qquad y \in [0, 1].$$

Since $T_2(y)$ deviates least from zero on $[0, 1]$ along all polynomials normalized in this way [8], it follows that $p_2(x) = x^2 + \frac{1}{8}$ is the best quadratic approximation to x on $[0, 1]$, and hence, being even, is also best for $[-1, 1]$. By (17), p_2 is a best 3-convex (and also a best 3-concave) approximation to $|x|$ on $[-1, 1]$.

(20) THEOREM. *If p_{n-1} is a bna to f then every bna to f agrees with p_{n-1} on an interval containing $n+1$ alternation points of $f - p_{n-1}$.*

Proof. Applying (10) with $g_1 = p_{n-1}$, we see that any other bna to g must agree with p_{n-1} on an alternant of length $n+1$. Thus $g - p_{n-1} - (f-g)$ has at least $n+1$ zeros, which by (2) implies that $g \equiv p_{n-1}$ on a subinterval of $[a, b]$ and $g \neq p_{n-1}$ elsewhere. Therefore $g \equiv p_{n-1}$ on a subinterval containing $n+1$ alternation points of $f - p_{n-1}$. ∎

The next theorem promises to be particularly useful in developing an algorithm for computing bna's

(21) THEOREM. *Let g be n-convex and suppose that $f - g$ has a negative alternant $x_0 < \cdots < x_n$ such that $[x_0, ..., x_n]\, g = 0$. Then g is a bna to f, and every other bna coincides with g on a subinterval of $[x_0, x_n]$ containing $n+1$ alternation points of $f - g$.*

Proof. We note first that by (2) $[x_0, ..., x_n]\, g = 0$ iff $g|_{[x_0, x_n]} \in \Pi_{n-1}$. If g_1 were n-convex with $\|f - g_1\| < \|f - g\|$, then $g_1 - g = (f-g) - (f-g_1)$ would have a negative oscillation at $x_0, ..., x_n$, a contradiction as $g_1 - g$ is n-convex on $[x_0, x_n]$. This shows that g is a bna. It also shows that g is best on the interval $[x_0, x_n]$; hence we may apply (20) to complete the theorem. ▮

(22) EXAMPLE. Let

$$\begin{aligned} f(x) &= x+1, & x &\in [-2, -1) \\ &= 0, & x &\in [-1, 1] \\ &= x-1, & x &\in (1, 2], \end{aligned}$$

i.e., $f(x) = x + 1 - (x+1)_+ + (x-1)_+$, where $x_+ = x$ if $x \leqslant 0$ and $x_+ = 0$ if $x < 0$. We seek a best 3-convex approximation to f on $[-2, 2]$. Note that f is neither 3-convex nor 3-concave since, although $f^{(3)} = 0$ except at 1 and -1, it is not differentiable at these points. The function

$$\begin{aligned} g(x) &= -(8x^2 + 8x + 1)/16, & x &\in [-2, -\tfrac{1}{2}) \\ &= (4x^3 - 3x)/16, & x &\in [-\tfrac{1}{2}, \tfrac{1}{2}] \\ &= (8x^2 - 8x + 1)/16, & x &\in (\tfrac{1}{2}, 2] \end{aligned}$$

is 3-convex since $g''(x)$ is nondecreasing, and one easily checks that $\{-2, -\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ is an alternant of length 8 for $f - g$, with $\|f - g\| = 1/16$. In particular, $f - g$ has a negative alternant of length 4 on $[\frac{1}{2}, 2]$, where $g \in \Pi_2$; hence by (13) g is a best 3-convex approximation to f.

3

In this section we briefly discuss polynomials and n-convex approximation to n-convex and n-concave functions.

The following lemma is compiled from results in [13].

(23) LEMMA. *If $f \in C[a, b]$ is n-convex and p_{n-1} is its best approximation from Π_{n-1} then*

(a) *the maximal length of an alternant for $f - p_{n-1}$ is $n+1$,*

(b) *$a, b \in \mathrm{crit}(f - p_{n-1})$ and $f(b) - p_{n-1}(b) \geqslant 0$,*

(c) *if $n \geqslant 2$ then a and b are isolated points of $\mathrm{crit}(f - p_{n-1})$, and*

(d) *if $n \geqslant 3$ then $\mathrm{crit}(f - p_{n-1})$ consists of precisely $n+1$ points forming a positively oriented alternant, including the endpoints.*

(24) LEMMA. *If* $f \in C[a, b] \backslash \Pi_{n-1}$ *is n-convex then its best approximation from* Π_n *is strictly n-convex (i.e.,* $a_n > 0$*).*

Proof. If p_n is the best approximation for f from Π_n then $f - p_n$ has an alternant of length at least $n+2$; hence there are points $a \leqslant t_0 < \cdots < t_n \leqslant b$ such that $(f - p_n)(t_i) = 0$ $(i = 0, ..., n)$. Thus $[t_0, ..., t_n](f - p_n) = 0$, and hence $0 \leqslant [t_0, ..., t_n]f = [t_0, ..., t_n]\, p_n = a_n$. If we suppose that $a_n = 0$ then $p_n \in \Pi_{n-1}$ so that $f - p_n$ is n-convex. But $f - p_n$ has at least $n+1$ sign changes, a contradiction; hence $a_n > 0$. ▮

(25) THEOREM [16]. *If f is n-concave then it has a bna in* Π_{n-1}.

Proof. If f is n-concave then by (24) its best approximation from Π_n is also n-concave; hence by (13) f has a best n-convex approximation in Π_{n-1}. ▮

(26) LEMMA (cf. [2]). *If f is nonincreasing on* $[a, b]$ *then* $p_0(x) = \frac{1}{2}(f(a) + f(b))$ *is its unique best nondecreasing approximation.*

Proof. The endpoints form an alternant of length 2 for $f - p_0$, with $\|f - p_0\| = \frac{1}{2}(f(a) - f(b))$. If g is a nondecreasing function with $\|f - g\| \leqslant \|f - p_0\|$ then necessarily $g(a) \geqslant p_0(a) = p_0(b) \geqslant g(b)$; hence g is constant and agrees with p_0 on $[a, b]$. ▮

(27) THEOREM. *If f is n-concave then* p_{n-1} *is its unique best n-convex approximation.*

Proof. If $f \in \Pi_{n-1}$ there is nothing to show. Otherwise by (24) $a_n < 0$; hence (13) implies that p_{n-1} is a bna. If g is another bna then from (20) $g \equiv p_{n-1}$ on an interval containing $n+1$ alternation points of $f - p_{n-1}$. If $n \geqslant 2$ then (23) implies that this interval is all of $[a, b]$, and we have uniqueness. For $n = 1$ uniqueness was proved in (26). ▮

We record that (27) follows from [6, Theorem 2.2], under slightly more restrictive conditions.

4

This section is devoted to a class of theorems that yield information on the "distance" from Π_{n-1} to a given continuous function, defined as

$$E_n(f) := \min\{\|f - \bar{p}\| : p \in \Pi_{n-1}\}.$$

Theorems of this type for n-convex functions were first proved by S. N. Bernstein, who always assumed that such functions were n-times differentiable. This is also the case in [8], where a variety of similar results are presented. No such assumption is made here.

(28) THEOREM (cf. [8]). *If* $f+g$ *and* $f-g$ *are n-convex then* $E_n(g) \leqslant E_n(f)$.

Proof. Let $\|g-q\| = E_n(g)$ and $\|f-p\| = E_n(f)$ for $p, q \in \Pi_{n-1}$, and let $a \leqslant x_0 < \cdots < x_n \leqslant b$ be an alternant for $g-q$. Set $h_- = (g-q) - (f-p) = p-q-(f-g)$ and $h_+ = (q-g)-(f-p) = p+q-(f+g)$. By assumption h_- and h_+ are both n-concave.

If we suppose that $\|f-p\| < \|g-q\| = \|q-g\|$, then $\operatorname{sgn} h_-(x_i) = \operatorname{sgn}(q-g)(x_i)$ $(i=0, ..., n)$; hence either h_+ or h_- has a positively oriented oscillation of length $n+1$, a contradiction as both are n-concave. ∎

(29) EXAMPLE. If $f^{(n)}(x) \geqslant n!$ on $[-1, 1]$ then $f(x) \pm x^n$ is n-convex; hence by (28) $E_n(f) \leqslant E_n(x^n) = \|T_n\| = 1/2^{n-1}$, where T_n is the monic Chebyshev polynomial of degree n on $[-1, 1]$.

(30) LEMMA [9]. *If* $f_1, f_2 \in C[a, b]$ *have similarly oriented alternants of length k and* $\|f_1\| = \|f_2\|$, *then* $f_1 - f_2$ *has at least k zeros, counting multiplicity up to two.*

(31) THEOREM. *If g and* $f-g$ *are n-convex then* $E_n(g) \leqslant E_n(f)$, *with equality iff* $f-g \in \Pi_{n-1}$.

Proof. As $f-g$ and $f+g = f-g+2g$ are n-convex, the inequality follows from (28). Now suppose that $\|f-p\| = E_n(g) = \|g-q\|$ for $p, q \in \Pi_{n-1}$. Since g and $f = f-g+g$ are n-convex, by (13) $f-p$ and $g-q$ have positively oriented alternants of length $n+1$; hence by (30) their difference has at least $n+1$ zeros. But $(f-g)-(p-q) = (f-p)-(g-q)$ is n-convex; hence from (2) it must vanish on a subinterval of $[a, b]$ and be nonzero elsewhere. However, (23) implies that $f-p$ and $g-q$ agree at the endpoints, from which it follows that $f-g = p-q \in \Pi_{n-1}$ on all of $[a, b]$. Conversely, if $f-g \in \Pi_{n-1}$ then $f = g + q_{n-1}$ for some $q_{n-1} \in \Pi_{n-1}$; hence $E_n(f) = E_n(g)$. ∎

(32) EXAMPLE. Applying (31) to the previous example we see that if $f^{(n)} \geqslant n!$ then $E_n(f) \leqslant 1/2^{n-1}$, with strict inequality unless $f^{(n)} = n!$, i.e., unless f is a monic polynomial of degree n.

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Note Added in Proof. After this work was completed, the author discovered that, in the paper of H. G. Burchard, Extremal Positive Splines with Applications to Interpolation and Approximation by Generalized Convex Functions, *Bull. Amer. Math. Soc.* **79** (1973), 959–964, a characterization theorem for best generalized convex approximation is announced, from which some of the results of this paper follow.

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